Deflection of symmetrical section beam in relation to stress

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Preview

The familiar equation for deflection of a "simple beam" in response to central loading force applied

$$y = \frac{FL^3}{48EI}$$

has the equivalent expression for deflection in relation to maximum stress state within the beam

$$y = \frac{\sigma L^2}{6EH}$$

which is useful.

The derivation

Symbols - all as familiarly used

I = Second Moment of Area
Z = Section Modulus
M = a Moment; a bending and/or turning force
F = a force (in Newtons; N)
L = length of the beam between supports
H = height of symmetrical section in direction it is being bent
σ = stress (in N/m^2)
y = beam bending deflection transverse to the length L dimension

Fundamental beam equations

Beam deflection *vs* force for a “simple beam”:

$$y = \frac{FL^3}{48EI}$$

Maximum moment (twist; torque) in a “simple beam”:

$$M = \frac{FL}{4}$$

The fundamental equation which for any beam combines the purely geometric property and the stress to give the moment in the beam:

$$M = \sigma Z$$

Precursor rearrangements

$Z = I/$half-height for a symmetrical section, so

$$Z = \frac{I}{H/2} = \frac{2I}{H}$$

Apply this to M

$$M = \sigma Z = \sigma \frac{2I}{H} = \frac{2\sigma I}{H}$$

Considered on its own and simply rearranged

$$M = \frac{FL}{4} \to F = \frac{4M}{L}$$

Substitutions

For F in $y = \frac{FL^3}{48EI}$ given $F = \frac{4M}{L}$:

$$y = \frac{4M}{L} \frac{L^3}{48EI} = \frac{ML^2}{12EI}$$

For M in above equation given $M = \frac{2\sigma I}{H}$:

$$y = \frac{2\sigma I}{H} \frac{L^2}{12EI} = \frac{\sigma L^2}{6EH}$$

noting this is the juncture at which I cross-cancels and disappears from this derived expression.

The objective is achieved, deriving

$$y = \frac{\sigma L^2}{6EH}$$

This equation will often be used in the transposed form

$$\sigma = \frac{6EHy}{L^2}$$

Significances

The Second Moment of Area I has cross-cancelled out of this derived expression. Much less information is needed to do this calculation of beam deflection *vs* (maximum) stress than is needed to do the beam calculations relating to force. Given calculation of I needs the cross-section fully described (shape; widths and heights, thicknesses, *etc*).
Whereas in the derived expression, the only characteristic of the cross-section needed is the height. Which can readily be measured for a beam already in service.

Useful applications of beam deflection *vs* stress

The equation $\sigma = \frac{6EHy}{L^2}$ can be applied to evaluate whether a beam already in service is bearing a load which is acceptable.
Here are the very practical steps:

- to get y, stretch a string from end to end along the pressed-against side of the beam and use a rule to measure the deflection of the beam away from straight - the gap between the string and the beam at the midpoint
- to get L, measure the length of the beam, typically with a tape measure
- to get H, measure the height of the beam, with rule or tape measure
- E the Elastic modulus of steel can be taken as 200GPa (210GPa is another common approximation used)
- apply the equation to calculate σ given y, L and H have been measured as described and E is almost invariant for steels and is known

- evaluate for the application whether the stress in the beam is acceptable and the safety factor is sufficient